The niche graphs of doubly partial orders

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Abstract

The competition graph of a doubly partial order is known to be an interval graph. The competition-common enemy graph of a doubly partial order is also known to be an interval graph unless it contains a cycle of length 4 as an induced subgraph. In this paper, we show that the niche graph of a doubly partial order is not necessarily an interval graph. In fact, we prove that, for each $n \geq 4$, there exists a doubly partial order whose niche graph contains an induced subgraph isomorphic to a cycle of length n. We also show that if the niche graph of a doubly partial order is triangle-free, then it is an interval graph.

Keywords: niche graph; doubly partial order; interval graph

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1 Introduction

Throughout this paper, all graphs and all digraphs are simple.

Given a digraph D, if (u, v) is an arc of D, we call v a *prey* of u and u a *predator* of v. The *competition graph* $C(D)$ of a digraph D is the graph which has the same vertex set as D and has an edge between vertices u and v if and only if there exists a common prey of u and v in D. The notion of competition graph is due to Cohen [3] and has arisen from ecology. Competition graphs also have applications in coding, radio transmission, and modelling of complex economic systems. (See [13] and [15] for a summary of these applications.) Since Cohen introduced the notion of competition graph, various variations have been defined and studied by many authors (see the survey articles by Kim [9] and Lundgren [11]). One of its variants, the *competition-common enemy graph* (or *CCE graph*) of a digraph D introduced by Scott [16] is the graph which has the same vertex set as D and has an edge between vertices u and v if and only if there exist both a common prey and a common predator of u and v in D. Another variant, the *niche graph* of a digraph D introduced by Cable *et al.* [1] is the graph which has the same vertex set as D and has an edge between vertices u and v if and only if there exists a common prey or a common predator of u and v in D.

A graph G is an *interval graph* if we can assign to each vertex v of G a real interval $J(v) \subset \mathbb{R}$ such that whenever $v \neq w$,

$$vw \in E \text{ if and only if } J(v) \cap J(w) \neq \emptyset.$$

The following theorem is a well-known characterization for interval graphs.

Theorem 1 ([7]). *A graph is an interval graph if and only if it is a chordal graph and it has no asteroidal triple.*

Cohen [3, 4] observed empirically that most competition graphs of acyclic digraphs representing food webs are interval graphs. Cohen's observation and the continued preponderance of examples that are interval graphs led to a large literature devoted to attempts to explain the observation and to study the properties of competition graphs. Roberts [14] showed that every graph can be made into the competition graph of an acyclic digraph by adding isolated vertices. (Add a vertex i_α corresponding to each edge $\alpha = \{a, b\}$ of G, and draw arcs from a and b to i_α.) He then asked for a characterization of acyclic digraphs whose competition graphs are interval graphs. The study of acyclic digraphs whose competition graphs are interval graphs led to several new problems and applications (see [5, 6, 10, 12]).
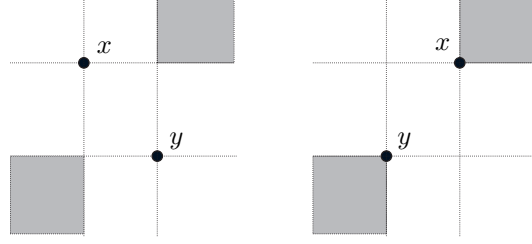
Figure 1: The region related to the adjacency of x and y

We introduce some notations for simplicity. A cycle of length n is denoted by C_n. For two vertices x and y in a graph G, we write $x \sim y$ in G when x and y are adjacent in G. For each point x in $\mathbb{R}^2$, we denote its first coordinate by x_1 and the second coordinate by x_2.

We define a partial order $\prec$ on $\mathbb{R}^2$ by

$$x \prec y \text{ if and only if } x_1 < y_1 \text{ and } x_2 < y_2.$$

For $x, y, z \in \mathbb{R}^2$, $x, y \prec z$ (resp. $x, y \succ z$) means $x \prec z$ and $y \prec z$ (resp. $x \succ z$ and $y \succ z$). For vertices x and y in $\mathbb{R}^2$, we write

$$
\begin{aligned}
x \searrow y & \qquad \text{if } x_1 \leq y_1 \text{ and } y_2 \leq x_2 \\
x \preceq y & \qquad \text{if } x_1 \leq y_1 \text{ and } x_2 \leq y_2.
\end{aligned}
$$

A digraph D is called a *doubly partial order* if there exists a finite subset V of $\mathbb{R}^2$ such that

$$V(D) = V \text{ and } A(D) = \{(v, x) \mid v, x \in V, x \prec v\}.$$

We may embed each of the competition graph, the CCE graph, and the niche graph of a doubly partial order D in $\mathbb{R}^2$ by locating each vertex at the same position as in D. We will always assume that D, its competition graph, CCE graph, and niche graph are embedded in $\mathbb{R}^2$ in natural way.

For two vertices x and y of a doubly partial order D, if there is a vertex of D in the region

$$
\begin{aligned}
& \{z \in \mathbb{R}^2 \mid z \prec (\min\{x_1, y_1\}, \min\{x_2, y_2\})\} \\
\cup \ & \{z \in \mathbb{R}^2 \mid z \succ (\max\{x_1, y_1\}, \max\{x_2, y_2\})\}
\end{aligned}
$$

(see Figure 1), then, by definition, x and y are adjacent in the niche graph of D.

The competition graph of a doubly partial order is an interval graph, and the CCE graph of a doubly partial order is also an interval graph if it is C_4-free:

Theorem 2 ([2]). *The competition graph of a doubly partial order is an interval graph.*

Theorem 3 ([8]). *The CCE graph of a doubly partial order is an interval graph unless it contains C_4 as an induced subgraph.*

It is natural to ask if another important variant of the competition graph, the niche graph, of a doubly partial order is an interval graph. In this paper, we show that for each $n \geq 4$, there is a doubly partial order whose niche graph contains an induced subgraph isomorphic to C_n, which implies that the niche graph of a doubly partial order is not necessarily an interval graph. Then we show that if the niche graph of a doubly partial order is triangle-free, then it is an interval graph.

2 Main results

We will show that the niche graph of a doubly partial order is not necessarily an interval graph. We first prove the following lemma.

For $c \in \mathbb{R}$, let $L_c := \{v \in \mathbb{R}^2 \mid v_1 + v_2 = c\}$ and $\mathbb{Z}^2 := \{v \in \mathbb{R}^2 \mid v_1, v_2 \in \mathbb{Z}\}$. Given a vertex v in a graph G, we denote by $\Gamma_G(v)$ the neighborhood of v in G.

Lemma 4. *Let V be a finite subset of $\mathbb{R}^2$ satisfying*

$$V \cap \mathbb{Z}^2 \subseteq L_c \cup L_{c+2} \quad and \quad V \setminus \mathbb{Z}^2 \subseteq \bigcup_{c < c' < c+2} L_{c'}$$

for some $c \in \mathbb{R}$. Suppose that $u_1 + 1 \neq v_1$ or $u_2 - 1 \neq v_2$ for two vertices u, v of $V \cap \mathbb{Z}^2$ with $u_1 \leq v_1$. Then $u \not\sim v$ in the niche graph of the doubly partial order D associated with V.

Proof. We prove by contradiction. Suppose that there exist two vertices $u, v \in V \cap \mathbb{Z}^2$ with $u_1 \leq v_1$ such that $u_1 + 1 \neq v_1$ or $u_2 - 1 \neq v_2$ but $u \sim v$ in the niche graph of D. Since $u \sim v$, there exists a vertex $a \in V$ such that either $a \prec u, v$ or $u, v \prec a$. Since $a \in V$,

$$c \leq a_1 + a_2 \leq c + 2. \tag{2.1}$$

Suppose that $\{u, v\} \not\subset L_c$ and $\{u, v\} \not\subset L_{c+2}$. Then either $u \in L_{c+2}$ and $v \in L_c$, or $u \in L_c$ and $v \in L_{c+2}$. This implies that

$$\min\{u_1 + u_2, v_1 + v_2\} = c \quad \text{and} \quad \max\{u_1 + u_2, v_1 + v_2\} = c + 2.$$

If $a \prec u, v$, then $a_1 + a_2 < \min\{u_1 + u_2, v_1 + v_2\} = c$, which contradicts (2.1). If $u, v \prec a$, then $a_1 + a_2 > \max\{u_1 + u_2, v_1 + v_2\} = c + 2$, which contradicts (2.1) again. Therefore either $\{u, v\} \subset L_c$ or $\{u, v\} \subset L_{c+2}$.

Now suppose that $\{u, v\} \subset L_c$. If $a \prec u, v$, then $a_1 + a_2 < u_1 + u_2 = c$, which is a contradiction to (2.1). Therefore it must hold that $u, v \prec a$. Then it is easy to check that

$$a_1 + a_2 > v_1 + u_2. \tag{2.2}$$

Since $u \neq v$ and $c = u_1 + u_2 = v_1 + v_2$, $u_1 \neq v_1$. By the assumption that $u_1 \leq v_1$, it is true that $u_1 < v_1$. Since $c = u_1 + u_2 = v_1 + v_2$, $u_2 > v_2$. In addition, from the assumption that $u_1 + 1 \neq v_1$ or $u_2 - 1 \neq v_2$, we have $v_1 - u_1 \geq 2$ or $u_2 - v_2 \geq 2$. If $v_1 - u_1 \geq 2$, then, by (2.2), $a_1 + a_2 > v_1 + u_2 \geq u_1 + u_2 + 2 = c + 2$, which contradicts (2.1). If $u_2 - v_2 \geq 2$, then, by (2.2), $a_1 + a_2 > v_1 + u_2 \geq v_1 + v_2 + 2 = c + 2$, which is a contradiction. Therefore it must hold that $\{u, v\} \subset L_{c+2}$.

If $u, v \prec a$, then $c + 2 = u_1 + u_2 < a_1 + a_2$, which is a contradiction to (2.1). Therefore it must hold that $a \prec u, v$. Then

$$a_1 + a_2 < u_1 + v_2. \tag{2.3}$$

Since $u \neq v$, $u_1 \leq v_1$, and $c + 2 = u_1 + u_2 = v_1 + v_2$, it is true that $u_1 < v_1$ and $v_2 > u_2$. Since $u_1 + 1 \neq v_1$ or $u_2 - 1 \neq v_2$, we have $v_1 - u_1 \geq 2$ or $u_2 - v_2 \geq 2$. If $v_1 - u_1 \geq 2$, then, by (2.3), $a_1 + a_2 < u_1 + v_2 \leq v_1 + v_2 - 2 = c$, which is a contradiction. If $u_2 - v_2 \geq 2$, then, by (2.3), $a_1 + a_2 < u_1 + v_2 \leq u_1 + u_2 - 2 = c$, which is a contradiction.

Hence u and v are not adjacent in the niche graph of D. $\qquad\square$

Theorem 5. *For any integer $n \geq 4$, there is a doubly partial order whose niche graph contains C_n as an induced subgraph.*

Proof. We construct a doubly partial order D_n for each integer $n \geq 4$. For any $(i, j) \in \mathbb{R}^2$, let $X_{(i,j)} := \{(i - 1, j - 1), (i, j), (i + 1, j + 1)\}$. For an integer k with $k \geq 2$, we define a finite subset W_k of $\mathbb{R}^2$ as follows:

$$W_k \cap \mathbb{Z}^2 := \{(i, k - 1 - i), (i + 1, k - i) \mid i = 0, 1, \ldots, k - 2\}$$

$$W_k \setminus \mathbb{Z}^2 := \{(i - \frac{1}{3}, k - i - \frac{1}{3}), (i + \frac{1}{3}, k - i + \frac{1}{3}) \mid i = 1, 2, \ldots, k - 2\} \quad (k \geq 3)$$

and $W_2 \setminus \mathbb{Z}^2 = \emptyset$. Let A_k be the sequence of vertices of $(W_k \cap \mathbb{Z}^2) \cup \{(0, k)\}$ listed as follows:

$$(k - 2, 1), (k - 3, 2), \ldots, (i, k - 1 - i), \ldots, (2, k - 3), (1, k - 2), (0, k - 1), \tag{$*$}$$

$$(0, k), (1, k), (2, k - 1), \ldots, (i + 1, k - i), \ldots, (k - 2, 3), (k - 1, 2).$$

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Let G_k be the niche graph of a doubly partial order associated with $X_{(0,k)} \cup W_k$. First, we will show that the sequence A_k is a path of length $2k - 2$ as an induced subgraph in G_k. In G_k, we can easily check the following:

(i) For $i = 0, 1, \ldots, k - 3$, the vertex $(i + 1 + \frac{1}{3}, k - 1 - i + \frac{1}{3})$ of $W_k \setminus \mathbb{Z}^2$ is a common predator of the $(k - 1 - i)$th vertex $(i, k - 1 - i)$ and the $(k - i)$th vertex $(i + 1, k - 2 - i)$;

(ii) For $i = 0, 1, \ldots, k - 3$, the vertex $(i + 1 - \frac{1}{3}, k - 1 - i - \frac{1}{3})$ of $W_k \setminus \mathbb{Z}^2$ is a common prey of the $(k + i + 1)$st vertex $(i + 1, k - i)$ and the $(k + i + 2)$nd vertex $(i + 2, k - 1 - i)$;

(iii) The vertex $(1, k + 1)$ is a common predator of the kth vertex $(0, k)$ and the $(k - 1)$st vertex $(0, k - 1)$;

(iv) The vertex $(-1, k - 1)$ is a common prey of the kth vertex $(0, k)$ and the $(k + 1)$st vertex $(1, k)$.

By (i) through (iv), the ith vertex and the jth vertex of the sequence A_k are adjacent in G_k if $|i - j| = 1$, and so A_k forms a path of length $2k - 2$ in G_k.

In addition, the sequence A_k is a path of length $2k - 2$ as an induced subgraph in G_k. To see why, we will show that the ith vertex and the jth vertex of A_k are not adjacent in G_k if $|i - j| \geq 2$. Take the ith vertex and the jth vertex of A_k with $|i - j| \geq 2$ and denote them by x and y. Suppose that $k = i$ or j. Then the kth vertex of A_k is $(0, k)$ and it is easy to check that

$$\Gamma_{G_k}((0, k)) = \{(1, k), (0, k - 1), (-1, k - 1), (1, k + 1)\}.$$

Since $(1, k)$ and $(0, k - 1)$ are the $(k + 1)$st vertex and $(k - 1)$st vertex of A_k, respectively, and $(-1, k - 1)$ and $(1, k + 1)$ are not vertices of A_k, we conclude that $x \nsim y$ in this case.

Suppose that $i \neq k$ and $j \neq k$. Without loss of generality, we may assume that $x_1 \leq y_1$. Note that W_k satisfies that

$$W_k \cap \mathbb{Z}^2 \subseteq L_{k-1} \cup L_{k+1} \quad \text{and} \quad W_k \setminus \mathbb{Z}^2 \subseteq \bigcup_{k-1 < c' < k+1} L_{c'}.$$

Since $|i - j| \geq 2$, $x_1 + 1 \neq y_1$ or $x_2 - 1 \neq y_2$ by the definition of A_k. Then, by Lemma 4, $x \nsim y$ in the niche graph of the doubly partial order associated with W_k. Therefore $x \nsim y$ in the subgraph of G_k induced by W_k. It remains to show that x and y have neither a common prey nor a common predator in $X_{(0,k)} = \{(-1, k - 1), (0, k), (1, k + 1)\}$. The set of predators

or prey of $(-1, k-1)$ in A_k is $\{(0,k), (1,k)\}$. These two vertices are kth and $(k+1)$st vertices of A_k and so $(-1, k-1)$ cannot be a common prey or a common predator of x and y. The set of predators or prey of $(0,k)$ in A_k is $\{(-1, k-1), (1, k+1)\}$ and so $(0,k)$ cannot be a common prey or a common predator of x and y. The set of predators or prey of $(1, k+1)$ in A_k is $\{(0,k), (0, k-1)\}$. These two vertices are kth and $(k-1)$st vertices of A_k and so $(-1, k-1)$ cannot be a common prey or a common predator of x and y. Hence we conclude that the ith vertex and the jth vertex of A_k are not adjacent in G_k if $|i - j| \geq 2$.

Now we are ready to give a construction of a doubly partial order D_n for each integer $n \geq 4$. Suppose that $n = 2k$ for some integer $k \geq 2$. Let

$$V_n := X_{(0,k)} \cup X_{(k-1,1)} \cup W_k$$

and D_n be the doubly partial order associated with V_n. We will show that the vertices of $(W_k \cap \mathbb{Z}^2) \cup \{(0,k), (k-1,1)\}$ form C_n without chord in the niche graph of D_n. See Figure 2 for an illustration. Let N_n be the niche graph of D_n.

Note that $X_{(k-1,1)} = \{(k-2, 0), (k-1, 1), (k, 2)\}$. Consider the sequence A_k defined in $(*)$. It is not difficult to check that none of vertices in $X_{(k-1,1)}$ can be a common prey or a common predator of two vertices of A_k. Thus by the previous argument, A_k forms a path as an induced subgraph of N_n. On the other hand, in the niche graph N_n of D_n, it can easily be checked that

$$\Gamma_{N_n}((k-2, 0)) = \{(k-1, 1), (k, 2), (k-1, 2)\};$$
$$\Gamma_{N_n}((k, 2)) = \{(k-1, 1), (k-2, 0), (k-2, 1)\};$$
$$\Gamma_{N_n}((k-1, 1)) = \{(k-2, 0), (k, 2), (k-2, 1), (k-1, 2)\}.$$

Thus, the vertices of A_k together with $(k-1, 1)$ form a cycle of length $2k = n$ as an induced subgraph.

Now we assume that n is an odd integer with $n \geq 5$. Then $n = 2k+1$ for some integer $k \geq 2$. Let

$$V_n := X_{(0,k)} \cup X_{(k+1,1)} \cup W_k$$

and D_n be the doubly partial order associated with V_n. See Figure 3 for an illustration. Note that $X_{(k+1,1)} = \{(k, 0), (k+1, 1), (k+2, 2)\}$.

Consider the sequence A_k defined in $(*)$. Then it is not hard to check that none of vertices in $X_{(k+1,1)}$ is a common prey or a common predator of two vertices of A_k. Thus, by the previous argument, A_k is a path as an induced subgraph of N_n.
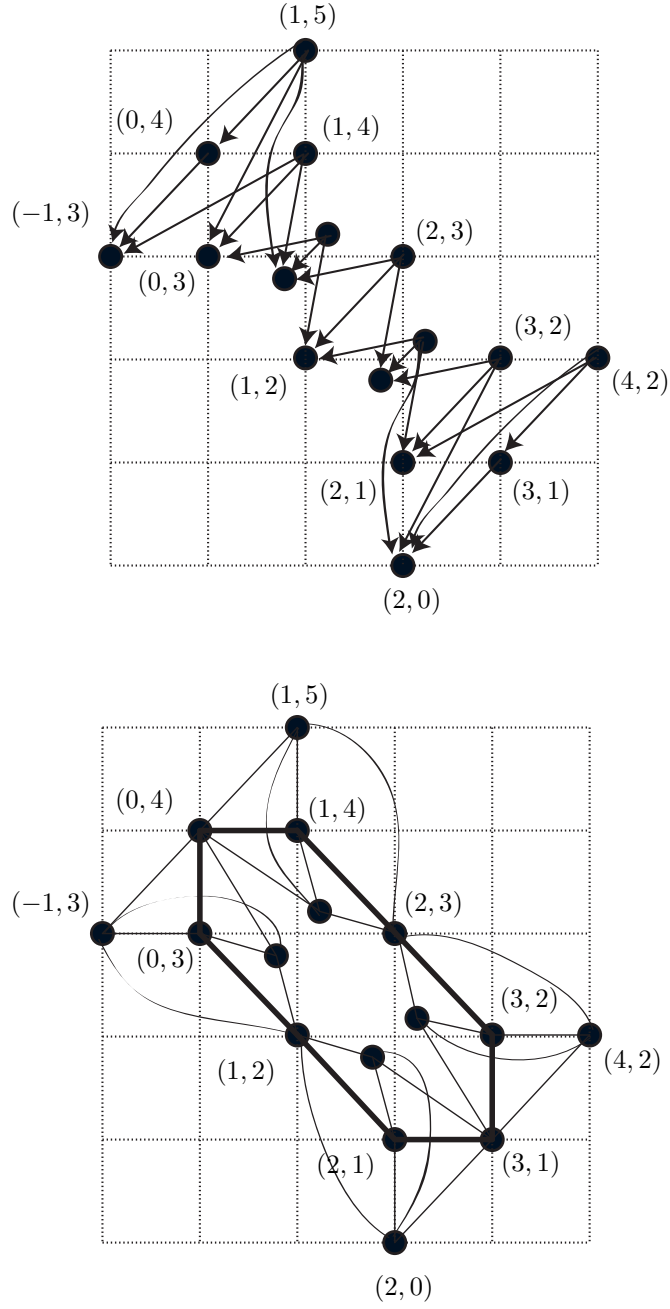
Figure 2: A doubly partial order D_8 and the niche graph of D_8. Note that the thick edges form a cycle of length 8 as an induced subgraph of the graph.
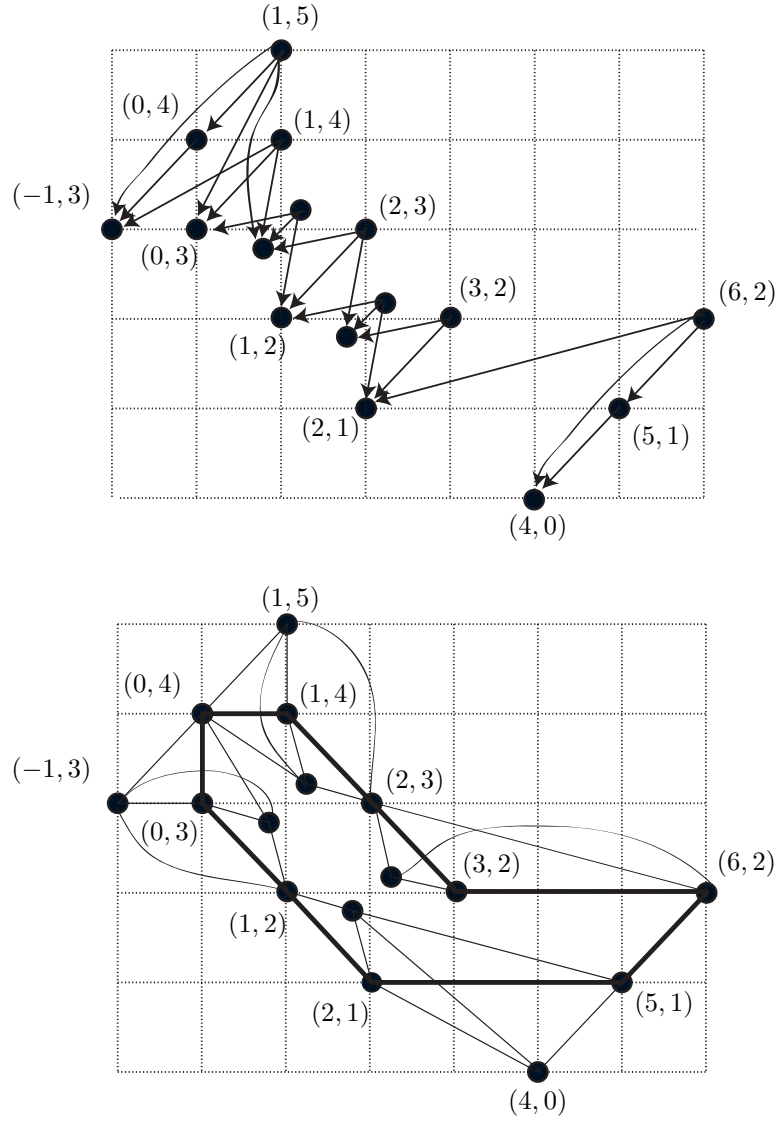
Figure 3: A doubly partial order D_9 and the niche graph of D_9. The thick edges form a cycle of length 9 as an induced subgraph of the graph.

It can easily be checked that

$$\Gamma_{N_n}((k,0)) = \{(k+1,1),(k-2,1)\};$$
$$\Gamma_{N_n}((k+1,1)) = \{(k,0),(k+2,2),(k-2,1)\};$$
$$\Gamma_{N_n}((k+2,2)) = \{(k+1,1),(k-1,2)\}.$$

Thus the first vertex $(k-2,1)$ of A_k is the only vertex in A_k adjacent to $(k+1,1)$. In addition, the $(2k-1)$st vertex $(k-1,2)$ of A_k are the only vertex in A_k adjacent to $(k+2,2)$. Since $(k+1,1)$ and $(k+2,2)$ are adjacent, the vertices of sequence A_k together with $(k+2,2)$ and $(k+1,1)$ form a cycle of length $2k+1 = n$ as an induced subgraph. Hence N_n contains C_n as an induced subgraph. $\square$

Theorems 1 and 5 tell us that the niche graph of a doubly partial order is not necessarily an interval graph. However if the niche graph of a doubly partial order is triangle-free, then it is an interval graph. To show that, we start with the following lemma:

Lemma 6. *Let D be a doubly partial order. Suppose that the niche graph G of D is triangle-free. Then if $x \sim y$, $y \sim z$ in G, and $x_1 \le z_1$, then $x \searrow y \searrow z$.*

Proof. Since $x \sim y$ and $y \sim z$ in G, there are vertices a and b such that either $a \prec x,y$ or $x,y \prec a$ and either $b \prec y,z$ or $y,z \prec b$. Suppose that $a \prec x,y$ and $y,z \prec b$. Then $a \prec y \prec b$ and so $a \prec b$. Therefore a is a common prey of x, y, and b, and so x, y and b form a triangle in G, which is a contradiction. Similarly, if $x,y \prec a$ and $b \prec y,z$, then we reach a contradiction. Hence either (1) $a \prec x,y$ and $b \prec y,z$, or (2) $x,y \prec a$ and $y,z \prec b$. In each case, we show that $x_1 \le y_1 \le z_1$. To show by contradiction, we consider two subcases (A) $x_1 > y_1$ and (B) $y_1 > z_1$ in each case.

Case 1. $a \prec x,y$ and $b \prec y,z$.

Subcase A. $y_1 < x_1$.

If $z_2 \le x_2$, then $b_1 < y_1 < x_1$ and $b_2 < z_2 \le x_2$ which imply that $b \prec x$. Then $b \prec x,y,z$ and so x, y, and z form a triangle in G, which is a contradiction. If $z_2 > x_2$, then $a_1 < y_1 \le x_1 \le z_1$ and $a_2 < x_2 < z_2$ which imply that $a \prec z$. Then $a \prec x,y,z$ and so x, y, and z form a triangle in G, which is a contradiction.

Subcase B. $z_1 < y_1$.

If $x_2 < y_2$, then $x \prec y$ and so $x,a,b \prec y$. Now suppose that $y_2 \le x_2$ and $y_2 \le z_2$. If $x_1 \le z_1$, then $a_1 < x_1 \le z_1$ and $a_2 < y_2 \le z_2$, which imply that $a \prec z$. Then $a \prec x,y,z$ and so x, y, and z form a triangle in G, which

is a contradiction. If $z_2 < y_2$, then $z \prec y$ and so $z, a, b \prec y$. Now suppose that $y_2 \leq x_2$ and $y_2 \leq z_2$. If $z_1 < x_1$, then $b_1 < z_1 < x_1$ and $b_2 < y_2 \leq x_2$, which imply that $b \prec x$. Then $b \prec x, y, z$ and so x, y, and z form a triangle in G, which is a contradiction.

Case 2. $x, y \prec a$ and $y, z \prec b$.

Subcase A. $y_1 < x_1$.

If $y_2 < x_2$, then $y \prec x$ and so $y \prec x, a, b$. Then x, a, and b form a triangle, which is a contradiction. If $y_2 < z_2$, then $y \prec z$ and so $y \prec z, a, b$. Then z, a, and b form a triangle, which is a contradiction. Now suppose that $x_2 \leq y_2$ and $z_2 \leq y_2$. If $x_1 \leq z_1$, then $x_1 \leq z_1 < b_1$ and $x_2 \leq y_2 < b_2$, which imply that $x \prec b$. Then $x, y, z \prec b$ and so x, y and z form a triangle in G, which is a contradiction. If $z_1 < x_1$, then $z_1 < x_1 < a_1$ and $z_2 \leq y_2 < a_2$, which imply that $z \prec a$. Then $x, y, z \prec a$ and so x, y and z form a triangle in G, which is a contradiction.

Subcase B. $z_1 < y_1$.

If $x_2 < z_2$, then $x_1 \leq y_1 < b_1$ and $x_2 < z_2 \leq b_2$ which imply that $x \prec b$. Then $x, y, z \prec b$ and so x, y, and z form a triangle in G, which is a contradiction. If $x_2 \geq z_2$, then $z_1 \leq y_1 < a_1$ and $z_2 \leq x_2 < a_2$ which imply that $z \prec a$. Then $x, y, z \prec a$ and so x, y, and z form a triangle in G, which is a contradiction.

Thus we can conclude that $x_1 \leq y_1 \leq z_1$ in each case. In addition, it cannot happen $x_1 = y_1 = z_1$. To see why, let c be an element of $\{a, b\}$ with smallest second component and d be the element of $\{a, b\} \setminus \{c\}$. Suppose that $a \prec x, y$ and $b \prec y, z$. Since $x_1 = y_1 = z_1$, we have $c \prec x, y, z$ and so x, y, and z form a triangle. Similarly, if $x, y \prec a$ and $y, z \prec b$, then $x, y, z \prec d$ and so x, y, z create a triangle. Therefore it holds that (1) $x_1 = y_1 < z_1$, (2) $x_1 < y_1 = z_1$, or (3) $x_1 < y_1 < z_1$. In the following, we show that $x_2 \geq y_2 \geq z_2$ in these three cases.

Case 1. $x_1 = y_1 < z_1$

Suppose that $x_2 < y_2$. If $x, y \prec a$ and $y, z \prec b$, then $x, y, z \prec b$. If $a \prec x, y$ and $b \prec y, z$, and $z_2 < x_2$, then $b \prec x, y, z$. If $a \prec x, y$ and $b \prec y, z$, and $z_2 \geq x_2$, then $a \prec x, y, z$. Therefore we reach a contradiction, and so it must hold that $x_2 \geq y_2$. Suppose that $y_2 < z_2$. If $a \prec x, y$ and $b \prec y, z$, then, since $b_1 < y_1 = x_1$ and $b_2 < y_2 \leq x_2$, we have $b \prec x, y, z$. If $x, y \prec a$ and $y, z \prec b$, then, since $y \prec a, b$ and $y \prec z$, we have $y \prec a, b, z$. Therefore we reach a contradiction, and so it must hold that $y_2 \geq z_2$. Thus $x_2 \geq y_2 \geq z_2$.

Case 2. $x_1 < y_1 = z_1$.

Suppose that $y_2 < z_2$. If $a \prec x, y$ and $b \prec y, z$, then $a \prec x, y, z$. If $x, y \prec a$ and $y, z \prec b$ and $z_2 \geq x_2$, then $x, y, z \prec b$. If $x, y \prec a$ and $y, z \prec b$

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and $z_2 < x_2$, then $x, y, z \prec a$. Therefore we reach a contradiction, and so it must hold that $y_2 \geq z_2$. Suppose that $x_2 < y_2$. If $x, y \prec a$ and $y, z \prec b$, then, since $z_1 = y_1 < a_1$ and $z_2 \leq y_2 < a_2$, we have $x, y, z \prec a$. If $a \prec x, y$ and $b \prec y, z$, then, since $a, b \prec y$ and $x \prec y$, we have $x, a, b \prec y$. Therefore we reach a contradiction, and so it must hold that $x_2 \geq y_2$. Thus $x_2 \geq y_2 \geq z_2$.

Case 3. $x_1 < y_1 < z_1$.

Suppose that $x_2 < y_2$. Then $x \prec y$. If $a \prec x, y$ and $b \prec y, z$, then $a, x, b \prec y$. If $x, y \prec a$ and $y, z \prec b$, then $x, y, z \prec b$. Therefore we reach a contradiction, and so $x_2 \geq y_2$. Suppose that $y_2 < z_2$. Then $y \prec z$. If $a \prec x, y$ and $b \prec y, z$, then $a, b, y \prec z$. If $x, y \prec a$ and $y, z \prec b$, then $y \prec a, b, z$. Therefore we reach a contradiction, and so $y_2 \geq z_2$. Thus $x_2 \geq y_2 \geq z_2$.

Hence we conclude that $x_2 \geq y_2 \geq z_2$ and so $x \searrow y \searrow z$. □

Theorem 7. *Let D be a doubly partial order. Suppose that the niche graph of D is a triangle-free graph. Then each component of the niche graph of D is a path.*

Proof. Let G be the niche graph of a doubly partial order D. First, we will show that G is a forest. Suppose that there is a cycle C of length n. We may assume that x is a vertex such that its first component x_1 is the minimum among those of vertices of C. Since G is triangle-free, $n \geq 4$ and so there exist 4 distinct vertices x, y, z, w such that $x \sim y$, $y \sim z$, $w \sim x$. Let u be the vertex of C such that $u \sim w$ and $u \neq x$. By the choice of x, $x_1 \leq u_1$ and $x_1 \leq z_1$. Then, since xwu and xyz are paths in G, $x \searrow w$ and $x \searrow y$ by Lemma 6. If $y_1 \geq w_1$, then, by Lemma 6, $w \searrow x$, which implies that $x = w$. If $y_1 < w_1$, then $y \searrow x$, which implies that $y = x$. Thus we reach a contradiction in either case. Hence G is a forest.

In the following, we will show that $\deg_G(v) \leq 2$ for any vertex v. Suppose that there is a vertex u such that $\deg_G(u) \geq 3$. Let x, y and z be three distinct neighbors of u. Without loss of generality, we may assume that $x_1 \leq y_1 \leq z_1$. Since xuy and yuz are paths in G, $x \searrow u \searrow y$ and $y \searrow u \searrow z$ by Lemma 6. Then $u \searrow y$ and $y \searrow u$ and so $y = u$, which is a contradiction. Hence each component of the niche graph of D is a path. □

By Theorem 1 and Theorem 7, the following theorem holds.

Theorem 8. *The niche graph of a doubly partial order is an interval graph unless it contains a triangle.*

3 Concluding remarks

We have shown that the niche graph of a doubly partial order is not necessarily an interval graph by constructing a doubly partial order whose niche graph contains a cycle an induced subgraph for each integer $n \geq 4$. Then we tried to find a doubly partial order such that its niche graph does not contain a cycle of length at least 4 as an induced subgraph and it is not an interval graph, but in vain. Accordingly, we would like to ask whether or not such a doubly partial order exists.

Eventually, it remains open to characterize doubly partial orders whose niche graphs are interval graphs.

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